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The Chebyshev Theory of a Variation of L_p $(1 < p < \infty)$ Approximation

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In this paper, we study a variation of best L_p approximation obtained by using a new "norm." We consider the questions of existence and uniqueness and also prove analogues of the essentials of the classical theory of best uniform approximation: characterization (Theorem 4), de La Vallée Poussin's bound (Theorem 5), and strong uniqueness (Theorem 7).

1. INTRODUCTION

After Pinkus and Shisha [1] proposed a new method of approximation using L_p-type "norms" (gauges), Y.-G. Shi [2] introduced another method of approximation in the case $p = 1$. This method maintains many essentials of the classical theory of best uniform approximation and has a distinct advantage over the corresponding one for L_1 best approximation, in that the unique best approximation is characterized by a remarkable geometric property. In this paper, we propose another L_p-type measure $\|\cdot\|^*$ $(1 < p < \infty)$ in terms of the technique of [2] in order to study the same questions of existence, uniqueness and characterization. In the special case $p = 1$, the measure $\|\cdot\|^*$ is the norm $\|\cdot\|$ defined in [2].

2. PRELIMINARIES

Let $C[a, b]$ be the class of real-valued functions continuous on $[a, b]$. For $f \in C[a, b]$ and $1 < p < \infty$, we define

$$\|f\|^* = \sup \left\{ \left| \int_c^d f\,|f|^{p-1}\,dx \right| : a \leqslant c \leqslant d \leqslant b \right\}. \tag{1}$$

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Let G be an n-dimensional subspace of $C[a, b]$. We consider the following problem: For a given $f \in C[a, b]$, find a $u \in G$ such that

$$\|f-u\|^* = \inf_{v \in G} \|f-v\|^*.$$

Such a polynomial u (if any) is said to be a best approximation to f from G.

Now we introduce the basic notations and definitions. Denote $X := \{I = (c, d) \colon I \subset [a, b]\}$. We adopt the convention that X contains the unique "zero" element $0 = (c, c)$. If $I = (c, d) \in X \setminus \{0\}$, we write $I^- = c$ and $I^+ = d$. In the following, we always assume that $f \in C[a, b]$. For ease of notation we set $f(I) := \int_I f\,|f|^{p-1}\,dx$, $X_f := \{I \in X\colon |f(I)| = \|f\|^*\}$, and $S_f(I) := \operatorname{sgn} f(I)$. With these notations (1) can be rewritten as

$$\|f\|^* = \sup_{I \in X} |f(I)|.$$

LEMMA 1. (a) *X is a compact set, $f(I)$ is a continuous function of I.*

(b) $\|f\|^* = \max_{I \in X} |f(I)|$.

(c) $\|f\|^* \leqslant \|f\|_p^p \leqslant (b-a)\,\|f\|_\infty^p$.

The proof is easy and is omitted.

Let $f, f_m \in C[a, b]$, $m = 1, 2, \ldots$, and let f_m converge to f, uniformly on $[a, b]$. From (c) of Lemma 1 it easily follows that

$$\lim_{m \to \infty} \|f_m - f\|^* = 0. \tag{2}$$

LEMMA 2. *If $f \in C[a, b]$, and C is a real number, then*

(a) $\|f\|^* = 0$ *if and only if* $f(x) = 0$ *for all* $x \in [a, b]$,

(b) $\|Cf\|^* = |C|^p\,\|f\|^*$,

(c) $\|\cdot\|^*$ *does not satisfy the triangle inequality.*

Proof. (a) and (b) are clear from the definition of $\|\cdot\|^*$.

(c) Let $f > 0$, $g > 0$. By the definition $\|f\|^* = \int_a^b f^p\,dx$, $\|g\|^* = \int_a^b g^p\,dx$ and $\|f+g\|^* = \int_a^b (f+g)^p\,dx$. Since $f^p + g^p < (f+g)^p$, this yields $\|f\|^* + \|g\|^* < \|f+g\|^*$.

LEMMA 3. *If $I \in X_f$ and $t > 0$ is sufficiently small, then*

(a) $I^-, I^+ \in Z(f) \cup \{a, b\}$, *where* $Z(f) = \{x \in [a, b]\colon f(x) = 0\}$,

(b) $f(I^- + t)\,f(I^- - t) \leqslant 0$, $f(I^+ - t)\,f(I^+ + t) \leqslant 0$,

(c) $S_f(I)\,f(I^- + t) \geqslant 0$, $S_f(I)\,f(I^+ - t) \geqslant 0$.

For the proof of (a) see [2, Lemma 1], and the rest is similar, too.

LEMMA 4. *Assume that $f, f_m \in C[a, b]$, $m = 1, 2, \ldots$ and f_m tends to f, uniformly on $[a, b]$. Then*

$$\|f\|^* = \lim_{m \to \infty} \|f_m\|^*. \tag{3}$$

Proof. At first, according to the Lebesgue Dominated Convergence Theorem, it follows that

$$\lim_{m \to \infty} f_m(I) = f(I), \qquad \forall I \in X. \tag{4}$$

Next, it is easy to check that

$$\max_{I \in X} |f_m(I)| - \max_{I \in X} |f(I)| \leqslant \max_{I \in X} [|f_m(I)| - |f(I)|].$$

Consequently from the property of the $\overline{\lim}$, the hypothesis of $f_m \to f$ and (4), we obtain that

$$\overline{\lim_{m \to \infty}} [\max_{I \in X} |f_m(I)| - \max_{I \in X} |f(I)|] \leqslant \lim_{m \to \infty} \max_{I \in X} [|f_m(I)| - |f(I)|] = 0. \tag{5}$$

Similary from

$$\max_{I \in X} |f_m(I)| - \max_{I \in X} |f(I)| \geqslant \min_{I \in X} [|f_m(I)| - |f(I)|],$$

it follows that

$$\underline{\lim}_{m \to \infty} [\max_{I \in X} |f_m(I)| - \max_{I \in X} |f(I)|] \geqslant \lim_{m \to \infty} \min_{I \in X} [|f_m(I)| - |f(I)|] = 0. \tag{6}$$

Combining (5) and (6) gives the result.

3. EXISTENCE

THEOREM 1. *Let $f \in C[a, b]$. There exists a $u \in G = \operatorname{span}\{g_1, \ldots, g_n\}$ for which*

$$\inf_{v \in G} \|f - v\|^* = \|f - u\|^*.$$

Proof. Set $\inf_{v \in G} \|f - v\|^* = C$. We may assume that $C > 0$. For $m = 1, 2, \ldots$, let $u_m = \sum_{k=1}^n a_k^{(m)} g_k \neq 0$ satisfy $\lim_{m \to \infty} \|f - u_m\|^* = C$ and let $\mu_m = \max\{|a_k^{(m)}|: 1 \leqslant k \leqslant n\} > 0$. We first show that μ_m is a bounded sequence. If this is not the case, then there exists a subsequence, again denoted by μ_m, which tends to ∞. By choosing a suitable subsequence, denoted by $\{\mu_{m_i}\}$, we may assume that, for $i=1, 2, ..., \mu_{m_i}=|a_{k_0}^{(m_i)}|$, with a fixed k_0, and that, for $k=1, ..., n$, $a_k^{(m_i)}/\mu_{m_i}$ converges, say, to a_k, $|a_k| \leqslant 1$, and $|a_{k_0}|=1$. Set $v=\sum_{k=1}^n a_k g_k$ and $v_m=\mu_m^{-1}(f-u_m)$, $m=1, 2, ...$. Then v_{m_i} tends uniformly to $-v$ on $[a, b]$. Since $v \neq 0$, $\|v\|^* > 0$. By (3), we have $\lim_{i\to\infty} \|v_{m_i}\|^* = \|v\|^* > 0$. However,

$$\lim_{i\to\infty} \|v_{m_i}\|^* = \lim_{i\to\infty} \|\mu_{m_i}^{-1}(f-u_{m_i})\|^* = \lim_{i\to\infty} \mu_{m_i}^{-p} \|f-u_{m_i}\|^* = 0.$$

This contradiction proves that μ_m is bounded.

Hence there are integers $1 \leqslant m_1 < m_2 < \cdots$ and reals $a_1, a_2, ..., a_n$ for which $\lim_{i\to\infty} a_k^{(m_i)} = a_k$, $k=1, ..., n$. Thus $\lim_{i\to\infty} u_{m_i} = u = \sum_{k=1}^n a_k g_k$ uniformly on $[a, b]$. By (3)

$$\|f-u\|^* = \lim_{i\to\infty} \|f-u_{m_i}\|^* = C.$$

The definition of C implies that

$$\inf_{v \in G} \|f-v\|^* = \|f-u\|^*.$$

4. Characterization

Definition 1. Let $f \neq 0$. An $I \in X_f$ is said to be a definite interval of f if there is no $J \subset I$ satisfying $f(J) = -f(I)$. The set of all definite intervals of f is denoted by X_f^*.

An $I \in X_f^*$ is said to be a maximal (resp. minimal) definite interval of f if there is no $J \supset I$ (resp. $J \subset I$) satisfying $J \in X_f^*$ and $J \neq I$. The set of all maximal (resp. minimal) definite intervals of f is denoted by X_f^M (resp. X_f^m).

Definition 2. $\{I_1, ..., I_m\} \subset X \setminus \{0\}$ is said to be weakly increasing if

(a) $I_i^- < I_{i+1}^-$, $I_i^+ < I_{i+1}^+$, $i=1, ..., m-1$.

(b) $I_i^+ < I_{i+2}^-$, $i=1, ..., m-2$.

If I and J are nonempty subintervals of $[a, b]$, $I < J := x < y$ for all $x \in I$ and $y \in J$.

$\{I_1, ..., I_m\} \subset X \setminus \{0\}$ is said to be increasing if $I_1 < \cdots < I_m$.

A system of extended intervals $I_1, ..., I_m$, i.e., $I_i \in X$ or $I_i = \{x\}$, $x \in [a, b]$, is said to be increasing if $I_1 < \cdots < I_m$.

Lemma 5. (a) *X_f^*, X_f^M, and X_f^m must exist.*

(b) *X_f^M is finite. Meanwhile $X_f^M = \{I_i\}_1^N$ with $I_1 \leqslant \cdots \leqslant I_N$ is weakly increasing and satisfies $f(I_{i+1}) = -f(I_i)$, $i=1, ..., N-1$.*

(c) *X_f^m is finite. Meanwhile $X_f^m = \{J_i\}_1^N$ with $J_1 < \cdots < J_N$ is increasing and satisfies $f(J_{i+1}) = -f(J_i)$, $i = 1, ..., N-1$.*

(d) *Card X_f^M = Card X_f^m, denoted by N_f. Furthermore if $X_f^M = \{I_1, ..., I_{N_f}\}$ and $X_f^m = \{J_1, ..., J_{N_f}\}$ are weakly increasing, then $J_i \subset I_i$, $f(I_i \setminus J_i) = 0$, $i = 1, ..., N_f$, and $J_i = (I_{i-1}^+, I_{i+1}^-)$, $i = 2, ..., N_f - 1$.*

The proof is similar to that of [2, Lemmas 4–7 and Theorems 2–4].

LEMMA 6. *Let $r, v \in C[a, b]$ and $I \in X$. Then*

$$\lim_{\lambda \to 0} \frac{(r+\lambda v)(I) - r(I)}{\lambda} = p \int_I v\, |r|^{p-1}.$$

Proof. By definition,

$$\lim_{\lambda \to 0} \frac{(r+\lambda v)(I) - r(I)}{\lambda} = \lim_{\lambda \to 0} \int_I \frac{(r+\lambda v)\,|r+\lambda v|^{p-1} - r\,|r|^{p-1}}{\lambda}. \tag{7}$$

Set $\Phi(\lambda) = (r+\lambda v)\,|r+\lambda v|^{p-1}$. Clearly $\Phi(\lambda)$ is a continuous function of λ and

$$\Phi'(\lambda) = \begin{cases} \dfrac{d(r+\lambda v)^p}{d\lambda} = pv(r+\lambda v)^{p-1} & r+\lambda v > 0, \\ 0, & r+\lambda v = 0, \\ -\dfrac{d(-r-\lambda v)^p}{d\lambda} = pv(-r-\lambda v)^{p-1} & r+\lambda v < 0. \end{cases}$$

$$= pv\,|r+\lambda v|^{p-1}.$$

Hence, there exists ξ, $|\xi| < |\lambda|$, satisfying

$$\left|\frac{\Phi(\lambda) - \Phi(0)}{\lambda}\right| = |\Phi'(\xi)| = pv\,|r+\xi v|^{p-1} \leqslant p\,|v|\,[|r| + |v|]^{p-1}$$

for $|\lambda| \leqslant 1$. This implies that $[\Phi(\lambda) - \Phi(0)]/\lambda$ is dominated by $p\,|v|\,[|r| + |v|]^{p-1}$. Thus according to the Lebesgue Dominated Convergence Theorem, we obtain that

$$\lim_{\lambda \to 0} \int_I \frac{(r+\lambda v)\,|r+\lambda v|^{p-1} - r\,|r|^{p-1}}{\lambda}$$

$$= \int_I \lim_{\lambda \to 0} \frac{(r+\lambda v)\,|r+\lambda v|^{p-1} - r\,|r|^{p-1}}{\lambda}$$

$$= \int_I \Phi'(0) = \int_I pv\,|r|^{p-1}$$

This combined with (7) completes the proof.

THEOREM 2. *Let* $G=\operatorname{span}\{g_1, \ldots, g_n\}$ *be an n-dimensional subspace of* $C[a, b]$, $f\in C[a, b]\backslash G$, $u\in G$, $r=f-u$, $S(I):=S_r(I)$. *Then a necessary condition for u to be a best approximation to f from G is that there do not exist* $v\in G$ *such that*

$$S(I)\int_I v\,|r|^{p-1}>0, \qquad \forall I\in X_r. \tag{8}$$

Proof. Suppose to the contrary that there is a $v\in G$ satisfying (8). At first for each $I\in X_r$, it follows from Lemma 6 that

$$\lim_{\lambda\to 0}\frac{S(I)(r+\lambda v)(I)-S(I)\,r(I)}{\lambda}=pS(I)\int_I v\,|r|^{p-1}>0.$$

Therefore $[S(I)(r+\lambda v)(I)-S(I)\,r(I)]/\lambda$ is a continuous function of λ and I and hence there exist sufficiently small positive numbers λ_I and ε_I such that

$$[S(I)(r+\lambda v)(I)-S(I)\,r(I)]/\lambda>0$$

and

$$S_{r-\lambda v}(J)=S_{r-\lambda v}(I)=S(I)$$

for all $0<\lambda<\lambda_I$ and $J\in\Delta(I, \varepsilon_I)=\{J\in X\colon |J^- - I^-|+|J^+ - I^+|\leqslant\varepsilon_I\}$. Thus we have

$$|(r-\lambda v)(J)|<S(I)\,r(J)\leqslant S(I)\,r(I)=\|r\|^*, \qquad J\in\Delta(I, \varepsilon_I),\ 0<\lambda\leqslant\lambda_I. \tag{9}$$

On the other hand, for each $I\notin X_r$, clearly $|r(I)|<\|r\|^*$. By the continuity of $(r-\lambda v)(I)$, there also exist sufficiently small positive numbers λ_I and ε_I, for which (9) does hold.

Secondly, it is clear that the compact set X is covered by an open covering $\bigcup_{I\in X}\Delta(I, \varepsilon_I)$. Hence we can collect a finite number of elements from the covering, denoted by $\{\Delta(I_i, \varepsilon_{I_i})\}_1^k$, for which

$$\bigcup_{i=1}^{k}\Delta(I_i, \varepsilon_{I_i})\supset X.$$

In other words, for each $I\in X$, there exists a j, $1\leqslant j\leqslant k$, such that $I\in\Delta(I_j, \varepsilon_{I_j})$. Set $\bar\lambda=\min\{\lambda_{I_1}, \ldots, \lambda_{I_k}\}$. Clearly $0<\bar\lambda\leqslant\lambda_{I_j}$. From (9) we see immediately that, for all $0<\lambda\leqslant\bar\lambda$, $|(r-\lambda v)(I)|<\|r\|^*$. This implies that, for all $0<\lambda\leqslant\bar\lambda$, we have

$$\|r-\lambda v\|^*<\|r\|^*.$$

This contradicts the hypothesis that u is a best approximation to f.

In the following, we shall be concerned with particular n-dimensional subspaces of $C[a, b]$, called QT-subspaces, considered by Y.-G. Shi [2], for which we can obtain a complete characterization of best approximation.

DEFINITION 3. A system of functions $\{g_1, ..., g_n\} \subset C[a, b]$ is said to be a quasi-Chebyshev system on $[a, b]$ (or a QT-system) and an n-dimensional subspace $G = \text{span}\{g_1, ..., g_n\}$ a QT-subspace if

$$D(I_1, ..., I_n) := \det \left| \int_{I_i} g_j \, dx \right|_{i, j=1}^{n} \neq 0,$$

whenever $\{I_i\}_1^n \subset X$ is increasing.

The following equivalent properties of a QT-system are from [2, Theorem 6].

THEOREM 3. *Let* $G = \text{span}\{g_1, ..., g_n\} \subset C[a, b]$. *Then the following statements are equivalent*:

(a) $\{g_1, ..., g_n\}$ *is a* QT-*system*.

(b) *For any weakly increasing intervals* $I_1, ..., I_n$, $D(I_1, ..., I_n) \neq 0$.

(c) $\{g_1, ..., g_n\}$ *is a nondegeneracy weak Chebyshev system on* $[a, b]$ (*see* [3]).

Before describing our result, let us note

LEMMA 7. *Let* $r, v \in C[a, b]$, $\{I_i\}_1^m \subset X$ *be weakly increasing and* $e = 1$ *or* -1, *fixed. Suppose*

$$(-1)^i e \int_{I_i} v \, |r|^{p-1} \geq 0, \qquad i = 1, ..., m.$$

Then

(a) *There exist* m *intervals* $J_1, ..., J_m$, $J_1 < \cdots < J_m$, *such that*

$$(-1)^i e \int_{J_i} v \, |r|^{p-1} \geq 0, \qquad i = 1, ..., m.$$

Furthermore, if $v(x) \not\equiv 0$ *on any nontrivial interval* $r(x) \not\equiv 0$ *on each* I_i, $\{J_i\}_1^m$ *may be chosen so that*

$$(-1)^i e \int_{J_i} v \, |r|^{p-1} > 0, \qquad i = 1, ..., m.$$

(b) *If* $m>1$, *there exist* $m-1$ *intervals* $K_1, \ldots, K_{m-1}$, $K_1 < \cdots < K_{m-1}$, *such that*

$$\int_{K_i} v\,|r|^{p-1} = 0, \qquad i=1, \ldots, m-1.$$

In addition, if $v(x) \not\equiv 0$ *on any nontrivial interval and* $r(x) \not\equiv 0$ *on each* I_i, *then* $\{K_i\}_1^{m-1}$, $K_1 < \cdots < K_{m-1}$ *may be chosen so that*

$$\int_{K_i} v\,|r|^{p-1} = 0, \qquad i=1, \ldots, m-1,$$

and $r(x) \not\equiv 0$ *on each* K_i.

The proof is similar to that of [2, Lemma 8].

LEMMA 8. *Let* $G = \mathrm{span}\{g_1, \ldots, g_n\}$ *be an n-dimensional QT-subspace of* $C[a, b]$, $\{I_i\}_1^n \subset X$ *be weakly increasing. If* $v \in G$, $r \in C[a, b]$ *are such that* $r(x) \not\equiv 0$ *on each* I_i, *and* $\int_{I_i} v\,|r|^{p-1} = 0$, $i=1, \ldots, n$, *then* $v=0$.

Proof. Suppose to the contrary that $v \neq 0$ and the conditions of the lemma hold. Take x, $\max\{I_n^-, I_{n-1}^+\} < x < I_n^+$, and denote $J_n = (I_n^-, x)$, $J_{n+1} = (x, I_n^+)$, and $J_i = I_i$, for $i=1, \ldots, n-1$. We see that $J_1, \ldots, J_{n+1}$ are also weakly increasing, $r(x) \not\equiv 0$ on each J_i, and $(-1)^i e \int_{J_i} v\,|r|^{p-1} \geqslant 0$, $i=1, \ldots, n+1$, where $e=1$ or -1, fixed. By Theorem 3, $\{g_1, \ldots, g_n\}$ is a nondegeneracy *WT*-system, hence v does not identically vanish on any nontrivial interval. By Lemma 7, there exist $\{K_i\}_1^n \subset X$, $K_1 < \cdots < K_n$, such that $r(x) \not\equiv 0$ on each K_i and $\int_{K_i} v\,|r|^{p-1} = 0$, for $i=1, \ldots, n$. This implies that v has at least one sign change on each K_i. Thus v has totally at least n sign changes. This is impossible because of G being a *WT*-subspace, which ends the proof.

COROLLARY 1. *Let* $G = \mathrm{span}\{g_1, \ldots, g_n\}$ *be an n-dimensional QT-subspace of* $C[a, b]$. *Let* $\{I_i\}_1^{n+1} \subset X$ *be weakly increasing and* $e=1$ *or* -1, *fixed. Let* $r \in C[a, b]$ *and* $r(x) \not\equiv 0$ *on each* I_i, $i=1, \ldots, n+1$. *If* $v \in G$ *satisfies*

$$(-1)^i e \int_{I_i} v\,|r|^{p-1} \geqslant 0, \qquad i=1, \ldots, n+1,$$

then $v=0$.

COROLLARY 2. *Let* $G = \mathrm{span}\{g_1, \ldots, g_n\}$ *be an n-dimensional QT-subspace of* $C[a, b]$. *Let* $\{I_i\}_1^{n-1} \subset X$ *be increasing and* $r \in C[a, b] \backslash \{0\}$. *Then there exists a nonzero polynomial* $v \in G$ *such that*

(a) $\int_{I_i} v\,|r|^{p-1} = 0, \qquad i=1, \ldots, n-1.$

(b) *v changes sign one time on each I_i, $i=1, ..., n$ and has no sign change in each (I_i^+, I_{i+1}^-), $i=0, 1, ..., n-1$, where $I_0^+=a$, $I_n^-=b$.*

Proof. Suppose r do not identically vanish on m intervals of $\{I_i\}_1^{n-1}$, $1\leq m\leq n-1$, denoted by $\{J_i\}_1^m$, and the rest by $\{J_i\}_{m+1}^{n-1}$. Note the following linear equations

$$\begin{aligned} \sum_{j=1}^{n} c_j \int_{J_i} g_j\,|r|^{p-1} &= 0, \qquad i=1, ..., m, \\ \sum_{j=1}^{n} c_j \int_{J_i} g_j &= 0, \qquad i=m+1, ..., n-1. \end{aligned} \tag{10}$$

Since $\{g_1, ..., g_n\}$ is a QT-system, it follows by using the theory of linear equations that (10) has a nonzero solution c_j, $j=1, ..., n$. Set $v=\sum_{j=1}^n c_j g_j$. It is easy to check that

$$\begin{aligned} \int_{J_i} v\,|r|^{p-1} &= 0, \qquad i=1, ..., m, \\ \int_{J_i} v &= 0, \qquad i=m+1, ..., n-1. \end{aligned} \tag{11}$$

Consequently, according to the above notation and the fact G is a nondegeneracy WT-subspace, we conclude from (11) that v satisfies (a) and (b).

LEMMA 9. *Let $G=\operatorname{span}\{g_1, ..., g_n\}$ be an n-dimensional QT-subspace of $C[a, b]$, $r\in C[a, b]$. Let a system of extended intervals $\{I_i\}_1^m=\{I_j'\}\cup\{x_k\}$ be increasing where $\{I_j'\}\subset X$ and $\{x_k\}\subset(a, b)$. Suppose $m<n$. Then there exists a nonzero polynomial $v\in G$ such that*

(a) $\int_{I_i} v\,|r|^{p-1}=0$, $i=1, ..., m$.

(b) *v changes sign on each I_i, $i=1, ..., m$. (If $I_i=x_k$, this means that v changes sign at x_k.)*

Proof. Put for $t>0$ sufficiently small

$$J_i=\begin{cases} (b-(n-i)t,\, b-(n-i-1)t), & i=m+1, ..., n-1 \text{ if } m<n-1 \\ (x_i-t, x_i+t) & \text{if } \quad I_i\in\{x_k\} \\ I_i\setminus\{(\bigcup[b-(n-i)t,\, b-(n-i-1)t]) & \\ \qquad\cup(\bigcup[x_i-t, x_i+t])\} & \text{if } \quad I_i\in\{I_j'\}. \end{cases}$$

We see that $\{J_i\}$ is also increasing if $t>0$ is sufficiently small. By Corollary 2, there exists a nonzero polynomial $v_t\in G$ such that

$\int_{J_i} v_t |r|^{p-1} = 0$, $i = 1, ..., n-1$, v_t changes sign one time on each J_i, $i = 1, ..., n-1$, and has no sign change in each interval (J_i^+, J_{i+1}^-), $i = 0, 1, ..., n-1$, where $J_0^+ = a$, $J_n^- = b$. The polynomial v_t can be assumed normalized in the sense that $\|v_t\|^* = 1$. Letting $t \to 0$, we select a limit polynomial $v \in G$ satisfying $\int_{I_i} v |r|^{p-1} = 0$, $i = 1, ..., m$, and for which v has no sign change in each interval (I_i^+, I_{i+1}^-), $i = 0, 1, ..., m$, where $I_0^+ = a$, $I_n^- = b$. It is not difficult to check that v changes sign on each I_i, $i = 1, ..., m$, and has exactly m sign changes. This completes the proof.

We now state our main result.

THEOREM 4. *Let $G = \text{span}\{g_1, ..., g_n\}$ be an n-dimensional QT-subspace of $C[a, b]$, $f \in C[a, b] \backslash G$, $u \in G$, $r = f - u$ and $S(I) := S_r(I)$. Then the following statements are equivalent.*

(a) *u is a best approximation to f from G.*

(b) *There does not exist a $v \in G$ such that*

$$S(I) \int_I v |r|^{p-1} > 0, \qquad \forall I \in X_r.$$

(c) *The origin of n space lies in the convex hull of the set $\{S(I)\hat{I}$: $I \in X_r\}$, where $\hat{I} = (\int_I g_1 |r|^{p-1}, ..., \int_I g_n |r|^{p-1})$.*

(d) $\max_{I \in X_r} S(I) \int_I v |r|^{p-1} \geq 0$, $\forall v \in G$.

(e) $\max_{I \in X_r} S(I) \int_I v |r|^{p-1} > 0$, $\forall v \in G \backslash \{0\}$.

(f) $N_r \geq n+1$.

Proof. Theorem 2 has shown that (a) $\Rightarrow$ (b), and (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) is clear by means of well-known arguments. We now show the other equivalences. Denote $N = N_r$ and $X_r^m = \{I_1, ..., I_N\}$ with $I_1 < \cdots < I_N$. Clearly, $r(x) \not\equiv 0$ on each I_i, $i = 1, ..., N$. Assume wihout loss of generality that $S(I_1) > 0$.

(b) $\Rightarrow$ (f). Suppose on the contrary that $N \leq n$. Put

$$K_i = \begin{cases} (I_i^-, I_{i+1}^+) & \text{if } i = \text{odd} \\ (I_i^+, I_{i+1}^-) & \text{if } i = \text{even and } I_i^+ < I_{i+1} \\ I_i^+ & \text{if } i = \text{even and } I_i^+ = I_{i+1}. \end{cases} \qquad (i = 1, ..., N-1).$$

Obviously the system of extended intervals $\{K_i\}_1^{N-1}$ is increasing. By Lemma 9, there is a nonzero polynomial $v \in G$ such that $\int_{K_i} v |r|^{p-1} = 0$, $i = 1, ..., N-1$, v changes sign on each K_i, $i = 1, ..., N-1$, and v has exactly $N-1$ sign changes on $[a, b]$. We assume that $\int_{I_1} v |r|^{p-1} > 0$ (taking $-v$ instead of v if necessary). Denote $K_0 = (a, K_1^-)$, $K_N = (K_{N-1}^+, b)$.

ASSERTION.

$$(-1)^{i+1}\int_{K_i^-}^{x} v\,|r|^{p-1}\geqslant 0, \qquad x\in \bar{K}_i,\ i>0. \tag{12}$$

$$(-1)^{i+1}\int_{x}^{K_i^+} v\,|r|^{p-1}\leqslant 0, \qquad x\in \bar{K}_i,\ i<N. \tag{13}$$

where $\bar{K}_i=[K_i^-, K_i^+]$.

Proof. If $K_i=\{x_k\}$, clearly, the equality in (12) and (13) holds. In the following we assume K_i is a nontrivial interval.

Case (i). $0<i<N$. In this case it follows from $\int_{K_i} v\,|r|^{p-1}=0$ that $\int_{K_i^-}^{x} v\,|r|^{p-1}=-\int_{x}^{K_i^+} v\,|r|^{p-1}$. Since $\int_{I_1} v\,|r|^{p-1}>0$ and v has exactly one sign change on K_i, we see immediately that

$$\left.\begin{aligned}(-1)^{i+1}\int_{K_i^-}^{x} v\,|r|^{p-1}&\geqslant 0,\\ (-1)^{i+1}\int_{x}^{K_i^+} v\,|r|^{p-1}&\leqslant 0,\end{aligned}\right\}\qquad x\in\bar{K}_i. \tag{14}$$

Case (ii). $i=0$. Since v changes sign once in (K_0^-, K_1^+), and that only in (K_1^-, K_1^+), it follows from the assumption of $\int_{I_1} v\,|r|^{p-1}>0$ that $\int_{x}^{K_0^+} v\,|r|^{p-1}\geqslant 0$, where $x\in\bar{K}_0$.

Case (iii). $i=N$. Applying the second inequality of (14) to the case $i=N-1$, we get that

$$(-1)^{N}\int_{x}^{K_{N-1}^+} v\,|r|^{p-1}\leqslant 0, \qquad x\in\bar{K}_{N-1}. \tag{15}$$

Next note that v changes sign once in (K_{N-1}^-, K_N^+), and that only in (K_{N-1}^-, K_{N-1}^+). Hence from (15) it is easy to infer that

$$(-1)^{N+1}\int_{K_N^-}^{x} v\,|r|^{p-1}\geqslant 0, \qquad x\in\bar{K}_N.$$

Now let $I\in X_r$ be arbitrary. Then I must contain an odd number of I_i's, say $I\supset(I_j\cup\cdots\cup I_{j+2k})$, where $j\geqslant 1$, $j+2k\leqslant N$, $k\geqslant 0$. Thus $I\supset(K_j\cup\cdots\cup K_{j+2k-1})$. Letting $L=(I^-, K_j^-)$ and $R=(K_{j+2k-1}^+, I^+)$, we have that

$$\begin{aligned}\int_I v\,|r|^{p-1}&=\int_L v\,|r|^{p-1}+\sum_{i=j}^{j+2k-1}\int_{K_i} v\,|r|^{p-1}+\int_R v\,|r|^{p-1}\\ &=\int_L v\,|r|^{p-1}+\int_R v\,|r|^{p-1}\end{aligned} \tag{16}$$

and

$$
\begin{aligned}
(-1)^j \int_L v\,|r|^{p-1} &\leqslant 0 \\
(-1)^{j+2k+1} \int_R v\,|r|^{p-1} &\geqslant 0
\end{aligned}
\tag{17}
$$

because of (12) and (13).

On the other hand, by the definition of L and R, it follows that

$$L = (I^{-}, K_{j-1}^{+}) \supset I_j \qquad \text{if} \quad j = \text{even}, \tag{18}$$

$$R = (K_{j+2k}, I^{+}) \supset I_{j+2k} \qquad \text{if} \quad j = \text{odd}. \tag{19}$$

Clearly, either (18) or (19) must occur. Then along with the condition that $r(x) \not\equiv 0$ on each I_i, $i = 1, ..., N$, we assert that r does not identically vanish on at least one of L and R. Thus at least one of the strict inequalities in (17) must hold. This combined with (16) and (17) gives that

$$(-1)^{j+1} \int_I v\,|r|^{p-1} > 0.$$

Next by the assumption of $S(I_1) > 0$, we get that $S(I) = S(I_j) = (-1)^{j+1} S(I_1) = (-1)^{j+1}$ and whence $S(I) \int_I v\,|r|^{p-1} > 0$, contradicting (b).

(f) $\Rightarrow$ (e). If not, let $v \in G \setminus \{0\}$ satisfy $\max_{I \in X_r} S(I) \int_I v\,|r|^{p-1} \leqslant 0$. Then $\max_{I \in X_r^m} S(I) \int_I v\,|r|^{p-1} \leqslant 0$ or $S(I_i) \int_{I_i} v\,|r|^{p-1} \leqslant 0$, $i = 1, ..., N$. Since $S(I_i) = (-1)^{i+1} S(I_1)$, $(-1)^i S(I_1) \int_{I_i} v\,|r|^{p-1} \geqslant 0$, $i = 1, ..., N$. Then because of $r(x) \not\equiv 0$ on each I_i, $i = 1, ..., N$, it follows from Corollary 1 that $v = 0$, a contradiction.

(e) $\Rightarrow$ (d). It is trivial to verify.

(f) $\Rightarrow$ (a). Suppose on the contrary that there exists a $v \in G \setminus \{0\}$ such that $\|r - v\|^* \leqslant \|r\|^*$. Whence for $\{I_i\}_1^N$,

$$S(I_j) \int_{I_j} (r - v)\,|r - v|^{p-1} < S(I_j) \int_{I_j} r\,|r|^{p-1}, \qquad j = 1, ..., N.$$

This implies that for each j, $1 \leqslant j \leqslant N$, there exists a point x_j, $x_j \in I_j$, such that

$$S(I_j)(r - v)(x_j) < S(I_j)\, r(x_j), \qquad j = 1, ..., N. \tag{20}$$

From (20) it is easy to conclude that $v \neq 0$ has at least n sign changes, which contradicts the fact that G is also a WT-subspace. This completes the proof.

We now provide an analogue of a fundamental result of de La Vallée Poussin.

THEOREM 5. *Let $G = \operatorname{span}\{g_1, ..., g_n\}$ be an n-dimensional QT-subspace of $C[a, b]$. Let $v \in G$ satisfy*

$$(-1)^i e \int_{I_i} (f-v)\,|f-v|^{p-1} \geqslant 0, \qquad i = 1, ..., n+1,$$

where $\{I_i\}_1^{n+1} \subset X$, $I_1 < \cdots < I_{n+1}$ and $e = 1$ or -1, fixed. Then

$$\inf_{u \in G} \|f-u\|^* \geqslant \min_{1 \leqslant i \leqslant n+1} \left| \int_{I_i} (f-v)\,|f-v|^{p-1} \right|.$$

The equality can occur if and only if v is a best approximation to f and $\{I_i\} \subset X_{f-v}$.

Proof. Letting $u \in G$ be a best approximation to f, $\|f-u\|^* \leqslant \min_{1 \leqslant i \leqslant n+1} |\int_{I_i} (f-v)\,|f-v|^{p-1}|$ implies that

$$\left|\int_{I_j} (f-u)\,|f-u|^{p-1}\right| \leqslant \min_{1 \leqslant i \leqslant n+1} \left|\int_{I_i} (f-v)\,|f-v|^{p-1}\right|, \qquad j = 1, ..., n+1.$$

If for every, i, $1 \leqslant i \leqslant n+1$, there exists a point x_i, $x_i \in I_i$, for which

$$(-1)^i e(f-u)(x_i) < (-1)^i e(f-v)(x_i), \qquad i = 1, ..., n+1,$$

then it is not difficult to see that this is impossible by using the fact that G is a WT-subspace. Therefore there must exist a j, $j \in \{1, ..., n+1\}$, such that

$$(-1)^j e(f-u)(x) = (-1)^j e(f-v)(x) \qquad \text{for all} \quad x \in I_j.$$

This combined with the fact again that G is a nondegeneracy WT-subspace yields that $v = u$ and of course, $\{I_i\} \subset X_{f-v}$. Conversely, if v is a best approximation to f and $\{I_i\} \subset X_{f-v}$, then equality occurs.

5. UNIQUENESS

THEOREM 6. *Let u be a best approximation from G to $f \in C[a, b]$. If G is a QT-subspace of $C[a, b]$, then u is unique.*

Proof. If $f \in G$, then $u = f$ is unique. Now suppose $f \notin G$. If possible, let $v \in G$ be another best approximation. Then for $X_{f-u}^m = \{I_i\}_1^{N_{f-u}}$, $I_1 < \cdots < I_{N_{f-u}}$, we have

$$(-1)^i e \int_{I_i} (f-u)\,|f-u|^{p-1} > 0,$$

where $e = -S_{f-u}(I_1)$ and

$$\|f-v\|^* = \|f-u\|^* = \min\left\{\left|\int_{I_i} (f-u)\,|f-u|^{p-1}\right| : 1 \leqslant i \leqslant N_{f-u}\right\}.$$

As in the proof of Theorem 5 we assert $u = v$.

LEMMA 10. *Let G be an n-dimensional subspace of $C[a, b]$. Then there exists a positive number C such that*

$$\|v\|^* \geqslant C\,\|v\|^p_\infty, \qquad \forall v \in G. \tag{21}$$

Proof. If $v = 0$, (21) is trivial. Otherwise, set

$$C = \inf_{v \in G \setminus \{0\}} \frac{\|v\|^*}{\|v\|^p_\infty},$$

we shall prove that $C > 0$. Suppose $C = 0$, then there exists a sequence $v_k \in G \setminus \{0\}$ such that

$$\frac{\|v_k\|^*}{\|v_k\|^p_\infty} \to 0, \qquad k \to \infty.$$

This means that for $u_k = v_k / \|v_k\|_\infty$ we have $\|u_k\|_\infty = 1$ and $\|u_k\|^* \to 0$, $k \to \infty$. Suppose without loss of generality that $u_k \to v$. Then $\|v\|_\infty = 1$ and by (3)

$$\|v\|^* = \lim_{k \to \infty} \|u_k\|^* = 0,$$

a contradiction.

Remark. This conclusion still holds if we take $G - f = \{v - f : v \in G\}$ instead of G.

With this conclusion we now present the strong uniqueness theorem.

THEOREM 7. *Let $G = \operatorname{span}\{g_1, ..., g_n\}$ be a QT-subspace, $u \in G$ is a best approximation to f. Then there exists a constant $\gamma > 0$ depending only on f such that for any $v \in G$*

$$\|f-v\|^* \geqslant \|f-u\|^* + \gamma\,\|u-v\|^q_\infty,$$

where $1 \leqslant q \leqslant p$.

Proof. If $f \in G$, it is trivial. Thus we assume $f \notin G$.

For any $v \neq u$, set

$$\gamma(v) = \frac{\|f - v\|^* - \|f - u\|^*}{\|u - v\|_r^q}.$$

We shall prove that $\gamma(v)$ has a positive lower bound. If this is not the case, then there exists a sequence $v_k \in G$, with $v_k \neq u$, $k = 1, 2, \ldots$, such that

$$\gamma(v_k) = \frac{\|f - v_k\|^* - \|f - u\|^*}{\|u - v_k\|_r^q} \to 0, \qquad k \to \infty.$$

We first prove that $\|u - v_k\|_r^q$ is a uniformly bounded sequence. In fact, if $\|u - v_k\|_r^q \to \infty$, $k \to \infty$, it follows from $\|f - v_k\|_r^q \to \infty$ and $\|f - v_k\|^* \geqslant C\|f - v_k\|_r^p \geqslant C\|f - v_k\|_r^q$ that

$$\varliminf_{k \to \infty} \gamma(v_k) \geqslant \varliminf_{k \to \infty} \frac{C\|f - v_k\|_r^q - \|f - u\|^*}{\|u - v_k\|_r^q} \geqslant C.$$

This contradiction proves that $\|u - v_k\|_r^q$ is bounded. Hence there is a positive constant M, for which $\|u - v_k\|_r^q \leqslant M$, $k = 1, 2, \ldots$. Without loss of generality assume that v_k converges uniformly to v, then it remains to check that v satisfies $\|u - v\|_r^q \leqslant M$, $v \in G$, and $\gamma(v) = 0$. Consequently applying Theorem 6 we obtain $v = u$.

Next set $C = \inf_{w \in G, \|w\|_r = 1} \max_{I \in X_{f-u}} S_{f-u}(I) \int_I pw\, |f - u|^{p-1}$. Clearly $C > 0$ from (e) of Theorem 4. Set $r = f - u$, $S_r(I) = S(I)$, and $v_k = u - \alpha_k$, where $\alpha_k \in G$ and α_k converges uniformly to 0 since v_k converges uniformly to $v = u$. Similar to the proof of Lemma 6, we can conclude that

$$\lim_{\alpha_k(x) \to 0} \frac{[(r + \alpha_k)\,|r + \alpha_k|^{p-1}](x) - [r\,|r|^{p-1}](x)}{\alpha_k(x)} = [p\,|r|^{p-1}](x)$$

for any $x \in [a, b]$.

Hence from the fact $|\alpha_k / \|\alpha_k\|_r| \leqslant 1$ we have

$$\lim_{\alpha_k \to 0} \Psi(\alpha_k) = \lim_{\alpha_k \to 0} \int_I \frac{\alpha_k}{\|\alpha_k\|_r} \left[\frac{(r + \alpha_k)\,|r + \alpha_k|^{p-1} - r\,|r|^{p-1}}{\alpha_k} - p\,|r|^{p-1} \right] = 0 \tag{22}$$

by using Lebesgue convergence Theorem.

Taking any $I \in X_r$ for which

$$pS(I) \int_I \frac{\alpha_k}{\|\alpha_k\|_r} |r|^{p-1} \geqslant C.$$

We calculate

$$\gamma(v_k) \geqslant \frac{S(I)\int_I (r+\alpha_k)\,|r+\alpha_k|^{p-1} - S(I)\int_I r\,|r|^{p-1}}{\|\alpha_k\|_I^q}$$

$$= \|\alpha_k\|_I^{1-q}\left[S(I)\,\bar{\Psi}(\alpha_k) + pS(I)\int_I \frac{\alpha_k}{\|\alpha_k\|_I}\,|r|^{p-1}\right]$$

$$\geqslant \|\alpha_k\|_I^{1-q}\,[S(I)\,\bar{\Psi}(\alpha_k) + C].$$

By (22),

$$\gamma(v) = \lim_{k\to\infty} \gamma(v_k) = \begin{cases} C & \text{if } q=1, \\ \infty & \text{if } 1<q\leqslant p, \end{cases}$$

a contradiction. This complete the proof.

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